

# Criticality in inhomogeneous magnetic systems: Application to quantum ferromagnets

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We consider a  $\phi^4$ -theory with a position-dependent distance from the critical point. One realization of this model is a classical ferromagnet subject to non-uniform mechanical stress. We find a sharp phase transition where the envelope of the local magnetization vanishes uniformly. The first-order transition in a quantum ferromagnet also remains sharp. The universal mechanism leading to a tricritical point in an itinerant quantum ferromagnet is suppressed, and in principle one can recover a quantum critical point with mean-field exponents. Observable consequences of these results are discussed.

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In standard phase transitions, such as the paramagnet-ferromagnet transition, or the liquid-gas transition, a homogeneous order parameter (OP; the magnetization in a magnet, or the density difference in a fluid) goes to zero as one crosses from the ordered phase into the disordered one. The OP may vanish continuously, as in the case of a magnet where the transition is second order, or discontinuously, as in the case of a fluid where the transition is first order except at the critical point. An external field may preclude a homogeneous OP. This happens for a fluid in a gravitational field, which produces a position-dependent density profile [1] and in some sense destroys the critical point (see below). Due to the weakness of gravity, this is a very small effect. This raises the question whether qualitatively similar, and maybe quantitatively larger, effects can be achieved in other systems if an external field induces an inhomogeneous OP.

We consider one such example, namely, a ferromagnet subject to mechanical stress. We will first discuss a classical Heisenberg magnet, and later generalize to quantum ferromagnets (FMs). Consider a metallic FM in the shape of a circular disk that is bent in the direction perpendicular to the disk plane. This leads to a position dependent mass density [2] and hence, in a metal, to a position dependent electron density and an inhomogeneous chemical potential  $\mu$ . The FM transition is described by a  $\phi^4$ -theory [3], and within a Stoner model the inhomogeneous  $\mu$  leads to a spatially dependent distance from criticality. Naively, this means that the system can be tuned to criticality only at special positions within the sample, not everywhere at the same time. One might thus expect the transition to become smeared. This is what appears to be found for the liquid-gas transition in a gravitational field [4, 5, 6], which does, however, present a physically different situation [7]. For the FM case we find that the transition remains sharp in a well-defined sense with mean-field critical behavior, even though the magnetization  $\mathbf{M}(\mathbf{x})$  is position dependent and hence “smeared” in some sense [8, 9]. One might ex-

pect that  $\mathbf{M}(\mathbf{x})$  is essentially restricted to a surface layer of fixed width, so that the dimensionality of the system is effectively reduced by one. This is not the case; we find (for a particular model of a sample of linear dimension  $L$ ) that  $\mathbf{M}(\mathbf{x})$  is essentially nonzero in a region of width  $L^{1/3}$ , so the support of the magnetization diverges in the thermodynamic limit,  $L \rightarrow \infty$ , albeit more slowly than  $L$ . This leads to unusual critical exponents for some spatially averaged observables.

Consider a flat circular disk sample of a metallic FM with thickness  $L$  (in the  $x$ -direction) and a radius that is some fixed multiple of  $L$ . A distortion of the disk in  $x$ -direction from a flat shape into a paraboloid leads to a strain tensor whose trace is a linear function of  $x$  [2], and hence to an electron density  $n(x) = n_0 + \text{const.} \times x$ . (This distortion must be achieved by bending, not, e.g., by grinding.) Within a Stoner model, the distance  $r$  from criticality depends linearly on the density of states, and hence on the cube root of  $n(x)$ . We consider a more general model where  $r$  varies as a power of  $x$ :

$$r(x) = r_0 + 2(x/L)^n. \quad (1)$$

Note that we take the prefactor of the  $x$ -dependent term to be of  $O(1)$  in order to demonstrate the qualitative effects of such a term. For a real bent plate the prefactor will be smaller, and we will give a semi-quantitative discussion below. Also note that  $r(L)$  is bounded as  $L \rightarrow \infty$ . This reflects a bending displacement proportional to  $L$  and ensures that a meaningful thermodynamic limit can be taken. Our model action is a  $\phi^4$ -theory with a spatially dependent mass given by  $r(x)$ ,

$$S = \int_V d\mathbf{x} \left[ \frac{r(x)}{2} \phi^2(\mathbf{x}) + \frac{c}{2} (\nabla \phi(\mathbf{x}))^2 + \frac{u}{4} (\phi^2(\mathbf{x}))^2 \right]. \quad (2)$$

The integration extends over a volume  $V \propto L^3$ , and  $c$  and  $u$  are constants. We emphasize that this model is rather general, and a magnet under stress is only one possible realization. We first treat this problem in a

saddle-point approximation and look for solutions of the form  $\phi(\mathbf{x}) = (0, 0, M(x))$ , with  $M(x)$  the inhomogeneous magnetization. The saddle-point equation then reads

$$c M''(x) = r(x) M(x) + u M^3(x) - H, \quad (3)$$

where we have added a magnetic field  $H$ . The physical solution of the ODE (3) must obey the boundary conditions  $M'(0) = M'(L) = 0$ .

While this ODE would be difficult to solve in closed form, we can obtain a great deal of information from asymptotic solutions and scaling considerations. For  $x \ll L$  one can neglect the  $x$ -dependence of  $r(x)$  and finds a solution in terms of the Jacobi integral  $\text{sn}(x)$ ; for  $x \approx L$  one can drop the  $M^3$  term and finds a solution in terms of Airy functions. A smeared transition would imply a nonzero magnetization for all parameter values [7]. This is physically not possible: for  $r_0 > 0$ ,  $r(x)$  is positive definite and the physical solution must be  $M(x) \equiv 0$ . We thus expect a sharp phase transition in the following sense: there exists a value  $r_0^c$  of  $r_0$  such that the envelope of the magnetization vanishes uniformly as  $r_0 \rightarrow r_0^c$  at  $H = 0$ , or as  $H \rightarrow 0$  at  $r_0 = r_0^c$ .

There are two explicit length scales in this problem: the zero of  $r(x)$ ,  $x_0 = L(-r_0/2)^{1/n}$ , and the bare correlation length  $\xi_0 = \sqrt{-c/r_0}$ . One expects the phase transition to occur when  $x_0 = \xi_0$  (apart from a factor of  $O(1)$ ). This condition leads to  $r_0^c = -2/\ell^{2n/(n+2)}$ , with  $\ell = L/\sqrt{c}$  a dimensionless system size. For  $r_0 = r_0^c$  the zero of  $r(x)$  is  $x_0^c = L/\ell^{2/(n+2)}$ . Now define  $y = x/x_0^c$ ,  $\mu(y) = \ell^{n/(n+2)} M(x)$ , and  $h = H \ell^{3n/(n+2)}$ .  $\mu$  obeys

$$\mu''(y) = \rho(y, r_0 \ell^{2n/(n+2)}) \mu(y) + u \mu^3(y) - h, \quad (4a)$$

where

$$\rho(y, z) = z + 2y^n. \quad (4b)$$

The solution of Eq. (4a) determines  $M(x)$  via the relation

$$M(x) = \ell^{-n/(n+2)} \mu\left(x/x_0^c; r_0 \ell^{2n/(n+2)}, H \ell^{3n/(n+2)}\right), \quad (5)$$

where we show the dependence of  $\mu$  on  $r_0$  and  $H$ .

Now consider the thermodynamic limit,  $\ell \rightarrow \infty$ . Since  $r(x)$  is bounded for all  $x$  (see Eq. (1)), we expect physical quantities in this limit to be independent of  $\ell$ . From Eq. (5), this requirement yields for the local magnetization a power-law prefactor times an envelope function,

$$M(x; r_0, H = 0) = r_0^{1/2} g_M^{r_0}(x/x_0^c), \quad (6a)$$

$$M(x; r_0 = 0, H) = H^{1/3} g_M^H(x/x_0^c). \quad (6b)$$

Similarly, for the envelope susceptibility  $\chi(x) = (\partial M(x)/\partial H)_{H=0}$  we have

$$\chi(x; r_0) = r_0^{-1} g_\chi(x/x_0^c), \quad (6c)$$

where  $g_M^{r_0}$ ,  $g_M^H$ , and  $g_\chi$  are scaling functions. We can further define a local specific heat  $C(x) = \partial^2 f(x)/\partial r_0^2$ ,

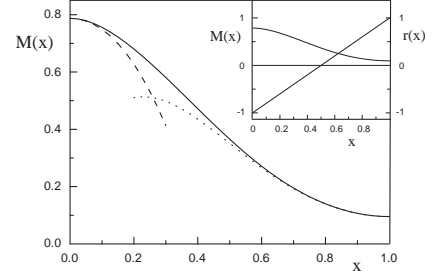


FIG. 1: The  $x$ -dependent magnetization (solid line) for  $r_0 = -1$ ,  $n = 1$ ,  $L = 1$ ,  $c = 1/25$ ,  $u = 1$ ,  $H = 0$ . The dashed and dotted lines represent asymptotic analytic solutions for  $x \ll L$  and  $x \approx L$ , respectively. The inset shows the magnetization together with the  $x$ -dependent mass function  $r(x)$ .

where  $f(x)$  is the free energy density which scales as  $M^4(x)$ . For the critical exponents  $\beta$ ,  $\delta$ ,  $\gamma$ , and  $\alpha$  defined by  $M(x) \propto r_0^\beta$ ,  $M(x) \propto H^{1/\delta}$ ,  $\chi(x) \propto r_0^{-\gamma}$ , and  $C(x) \propto r_0^{-\alpha}$  this implies

$$\beta = 1/2, \quad \delta = 3, \quad \gamma = 1, \quad \alpha = 0. \quad (7a)$$

Finally, we determine the exponents  $\nu$  and  $\eta$ . The magnetization depends on  $r_0$  only through the combination  $r_0 \ell^{2n/(n+2)}$ . If we identify the diverging length scale  $\xi$  that characterizes the phase transition with  $c^{1/2} \ell^{n/(n+2)}$  we have  $r_0 \sim \xi^{-2}$ , or  $\xi \sim r_0^{-\nu}$  with  $\nu = 1/2$ . Furthermore, the inverse susceptibility determines the exponent  $\eta$  via  $\chi^{-1} \propto r_0 \sim \xi^{-2} \equiv \xi^{-2+\eta}$  with  $\eta = 0$ . We thus have

$$\nu = 1/2, \quad \eta = 0. \quad (7b)$$

$\nu$  and  $\eta$  defined in this way are finite-size scaling exponents, and  $\nu$  does not represent the divergence of a coherence length defined via the spatial decay of the two-point correlation function. The latter remains finite even at the transition, as it does in the case of the liquid-gas transition in a gravitational field [6]. There thus is also a correlation length exponent  $\nu$  that is equal to zero. There is a further ambiguity within the framework of finite-size scaling: defining  $\xi$ , for instance, as  $\xi = c^{1/2} \ell$  would lead to different values of  $\nu$  and  $\eta$ . The mean-field values given above result from what in some sense is the most natural choice for  $\xi$ . However, all choices for  $\xi$  preserve the exponent relation  $\nu(2 - \eta) = \gamma$ . The Essam-Fisher relation,  $\alpha + 2\beta + \gamma = 2$ , is also fulfilled.

We have corroborated and augmented these results by solving Eq. (3) numerically with the boundary condition  $M'(0) = 0$ , and  $M(0)$  chosen such that the mean-field free energy is minimized. Fig. 1 shows  $M(x)$  for the case of a linearly  $x$ -dependent mass,  $n = 1$ .  $M(x)$  is large in the region where  $r(x) < 0$  and small in the region where  $r(x) > 0$ , as one would expect. The asymptotic solutions mentioned above are also shown in Fig. 1.

With increasing  $r_0$ , the magnetization decreases for all  $x$ , see Fig. 2, and it vanishes uniformly when  $r_0$  reaches

the critical value  $r_0^c$ . For  $r_0 > r_0^c$  the physical solution of Eq. (3) is  $M(x) \equiv 0$ . As  $r_0 \rightarrow r_0^c$  from below, the envelope of the magnetization vanishes as  $|r_0 - r_0^c|^{1/2}$ . For the value  $M(x=0)$  this is demonstrated in the inset in Fig. 2. In a magnetic field the magnetization vanishes uniformly as  $H^{1/3}$  for  $H \rightarrow 0$  at  $r_0 = r_0^c$ , see Fig. 3. These results show that there is a sharp phase transition: the envelope of the magnetization vanishes uniformly as  $r_0 \rightarrow r_0^c$  at  $H = 0$ , or as  $H \rightarrow 0$  at  $r_0 = r_0^c$ . The order-parameter critical exponents have mean-field values:  $\beta = 1/2$  and  $\delta = 3$ . The numerics suggest that the mean-field critical behavior also holds for finite  $L$ . We have found Eqs. (6a, 6b) to hold for values of  $L^2/c$  as small as 16.

We now consider spatially averaged observables (denoted by an overbar) rather than local quantities. Consider an averaged magnetization  $\bar{M} = (1/L) \int_0^L dx M(x)$ . Since  $M(x)$  is essentially nonzero only on the interval  $x \in [0, x_0^c]$ , the upper limit of the integral is essentially  $x_0^c \propto \ell^{n/(n+2)}$ . Corrections to this approximation are exponentially small.  $\bar{M}$  obeys, instead of Eq. (5),

$$\bar{M} = \ell^{-1} f_M(r_0 \ell^{2n/(n+2)}, H \ell^{3n/(n+2)}), \quad (8a)$$

$$f_M(u, v) = \int_0^1 dy \mu(y; u, v). \quad (8b)$$

Demanding again that observables are independent of  $\ell$  for  $\ell \rightarrow \infty$ , this leads to exponents  $\bar{\beta} = (n+2)/2n$  and  $\bar{\delta} = 3n/(n+2)$ . Analogous considerations for the other observables we have considered yield the following set of exponents for spatially averaged quantities,

$$\bar{\beta} = \frac{n+2}{2n}, \quad \bar{\delta} = \frac{3n}{n+2}, \quad \bar{\gamma} = \frac{n-1}{n}, \quad \bar{\alpha} = \frac{-1}{n}. \quad (9a)$$

From the averaged susceptibility  $\bar{\chi} = (\partial \bar{M} / \partial H)_{H=0}$  we find the exponent  $\bar{\eta}$ , and  $\nu$  is unchanged since  $\bar{M}$  depends on the same combination of  $r_0$  and  $\ell$  as  $M(x)$ ,

$$\bar{\eta} = 2/n, \quad \bar{\nu} = 1/2. \quad (9b)$$

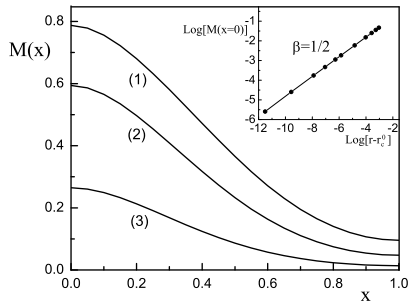


FIG. 2: The  $x$ -dependent magnetization for  $r_0 = -1$  (1),  $r_0 = -0.8$  (2), and  $r_0 = -0.6$  (3). All other parameter values are the same as in Fig. 1, which lead to  $r_0^c = -0.55213$ . The inset shows a log-log plot of  $M(x=0)$  vs.  $r_0 - r_0^c$ , with the solid line representing a power law with an exponent of  $1/2$ . See the text for additional information.

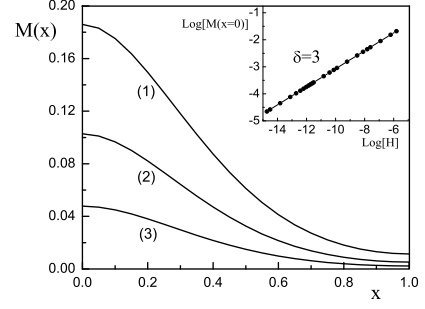


FIG. 3: The  $x$ -dependent magnetization for  $r_0 = r_0^c$  and  $H = 0.003$  (1),  $H = 0.0005$  (2), and  $H = 0.00005$  (3). All other parameter values are the same as in Fig. 1. The inset shows a log-log plot of  $M(x=0)$  vs  $H$ , with the solid line representing a power law with an exponent of  $1/3$ .

These exponents for the averaged quantities satisfy again the relations  $\bar{\alpha} + 2\bar{\beta} + \bar{\gamma} = 2$  and  $\bar{\nu}(2 - \bar{\eta}) = \bar{\gamma}$ .

The exponent values depend on how the averaged quantities are defined, and hence on what exactly is being measured. For instance, if one defined  $\bar{M}$  as  $(1/c^{1/3}L^{1/3}) \int_0^L dx M(x)$  to account for the fact that the magnetization is essentially nonzero only for  $x < x_0^c \propto \ell^{1/3}$ , one would find mean-field values for all exponents.

We now apply these results to quantum FMs. It is observed that the transition in itinerant FMs at sufficiently low temperatures  $T$  is always first order [10]. This has been explained in terms of fluctuation effects due to the coupling of the OP fluctuations to soft particle-hole excitations in metallic FMs. As a result, the free energy in a mean-field approximation has the form [11]

$$F = \frac{r}{2} M^2 + \frac{v}{4} M^4 \ln(M^2 + T^2) + \frac{u}{4} M^4 + O(M^6), \quad (10)$$

with  $v > 0$ . The  $v$ -term is negative, which leads to a first-order transition at  $T = 0$ , and to a tricritical point at a temperature  $T_{tc} = \exp(-u/2v)$  [12].

A position dependent chemical potential in the regime where the transition is first order has two effects: (1) It lowers the tricritical temperature  $T_{tc}$ , and a sufficiently strong space dependence restores a quantum critical point; (2) it leads to an inhomogeneous magnetization as in the case of classical magnets discussed above.

To quantify the first effect we estimate the prefactor of the space dependent term in  $\mu$ . For a displacement at the edge of the plate equal to  $0.01L$  (conservatively; more severe bending may be possible) one finds a density variation of about  $0.01n_0$ , and, instead of Eq. (1),  $r(x) = r_0 + 2\rho x/L$  with  $\rho = O(0.01)$ . Suppressing factors of  $O(1)$ , the chemical potential is  $\mu(x) = \epsilon_F(1 + \rho x/L)$ . The effects are smaller by a factor of about  $10^2$  compared to those shown in our figures, but still large compared to those of gravity on a fluid. To estimate the effect on  $T_{tc}$  we recall that the  $v$ -term in the free energy is due

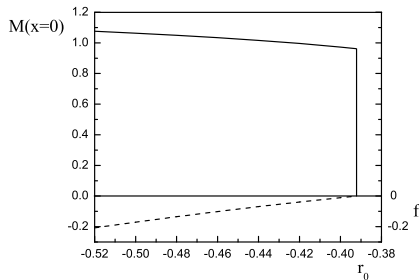


FIG. 4:  $M(x=0)$  vs.  $r_0$  (solid line, left scale) as obtained from the solution of Eq. 11 for parameter values  $L = 1$ ,  $c = 1/25$ ,  $u = -1$ ,  $w = 1$ , which lead to  $r_0^1 = -0.39216$ . Also shown is the free energy density  $f$  (dashed line, right scale).

to a soft propagator  $P = 1/(\omega + kv_F)$  [13], with  $\omega$  ( $k$ ) the frequency (wave number). A position dependent  $\mu$  causes  $P$  to acquire a mass or cutoff frequency  $\omega_c$  proportional to the prefactor of the  $x$ -dependency in  $\mu(x)$ ,  $\omega_c \approx \rho \epsilon_F / \hbar L k_F$ .  $T_{tc}$  is lowered by roughly  $\hbar \omega_c / k_B$ . With  $\rho$  as above, and typical values for all other quantities, one finds a very small suppression  $\delta T_{tc} = O(10^{-6} \text{ K})$ . It is interesting, however, that a sufficiently strong spatial dependence of  $\mu$  will eventually destroy the mechanism that leads to a first-order quantum phase transition, as does quenched disorder [13].

To study the second effect we thus assume that the transition is still first order at low temperature. In this case the logarithm in Eq. (10) can be expanded, and the Landau free energy is adequately represented by a power series with a negative  $M^4$  term and a positive  $M^6$  term. Generalizing to a non-homogeneous situation we then obtain the following ODE for the OP in the quantum case,

$$c M''(x) = r(x) M(x) + u M^3(x) + w M^5(x), \quad (11)$$

which replaces Eq. (3). Here  $w > 0$ ,  $u < 0$ , and  $r(x)$  is given by Eq. (1). The resulting  $M(x)$  curves are very similar to those in Fig. 2, but now  $M(x)$  goes to zero discontinuously at a critical value  $r_0^1$  of  $r_0$ , see Fig. 4.

We now consider the validity of our mean-field treatment. If the quantum critical point is restored by a (hypothetical) large position dependence of  $\mu$ , the quantum critical behavior will be mean-field like since the spatial dependence of  $\mu$  suppresses the mechanism that causes a first-order transition in clean quantum FMs, and non-mean-field critical behavior in disordered ones [13]. For the classical transition it is possible that the mean-field critical behavior will also be exact. This is because the two-point correlation function is always finite, and hence there are no divergencies in simple perturbation theory. However, the divergence of the envelope susceptibility, Eq. (6c), shows that there are fluctuations that may influence the critical behavior. Fluctuations of the elastic deformation should also be considered [14, 15]. These points require additional investigation.

In summary, we have considered a model for a metallic FM with a position dependent electron density or chemical potential. This can be realized by mechanically stressing the sample. The phase transition remains sharp even though the electron density is not homogeneous, and we have given critical exponents for both local and spatially averaged observables. In the quantum case, the tricritical temperature is lowered, although for realistic stresses this is a small effect. If a stronger position dependence can be realized (e.g., by means of optical lattices), this will eventually suppress the tricritical point, restoring a quantum critical point with mean-field critical behavior.

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  - [8] The  $\lambda$ -transition in He 4 is described by an XY-model with a spatially dependent mass. The prevailing view is that in this case, too, there is no sharp transition [5]. However, the actual observations are consistent with a sharp phase transition in the sense described here.
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